$X_1, \dots, X_n$ n v. a. indipendenti
tutte con legge $N(\mu, \sigma^2)$
Allora le v. a.

$$\frac{\bar{X}-\mu}{\sigma}\sqrt{n} = \boxed{\frac{X_1 + \dots + X_n - n\mu}{\sigma\sqrt{n}}}$$ ha legge $N(0,1)$

dove $\bar{X} = \frac{X_1 + \dots + X_n}{n}$

(CLT)

Teorema Limite Centrale (TLC)

Sia $X_1, X_2, X_3, \dots$ una successione di v. a. indipendenti, tutte con la stessa legge (i.i.d. = independent identically distributed), aventi media μ e varianza σ^2 finite

$$E[X] = \sum x_i \, p(x_i)$$

$X_1 \quad \mu_1 \quad X_2 \quad \mu_2 \quad \dots$

Consideriamo le v.a.

$$\boxed{S_n^* = \frac{X_1 + \dots + X_n - n\mu}{\sigma\sqrt{n}}}$$

Allora, $\forall x \in \mathbb{R}$

$$\lim_{n \to \infty} P(S_n^* \le x) = \Phi(x)$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

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$$S_n = X_1 + \dots + X_n$$

$$P(S_n \leq t)$$

$$X_i = \begin{cases} 1 & p \\ 0 & 1-p \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i \sim B(n,p)$$

$$\left.\begin{array}{l} X \sim \Pi_\lambda \\ Y \sim \Pi_\mu \end{array}\right\} \text{indip}$$

$$X+Y \sim \Pi_{\lambda+\mu}$$

$$X_i \sim \Pi_\lambda$$

$$S_n \sim \Pi_{n\lambda}$$

$$X_i \sim \mathcal{E}(\lambda) = \Gamma(1,\lambda)$$

$$S_n \sim \Gamma(n,\lambda)$$

$$S_n = X_1 + \dots + X_n$$

X_i indip e tutte con la stessa legge

$$P(S_n \le t) =$$

$\mu = E[X_1]$

$\sigma^2 = \mathrm{Var}\, X_1$

$$= P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le \overbrace{\frac{t - n\mu}{\sigma\sqrt{n}}}^{x} \right) =$$

$$= P(S_n^* \le x) \xrightarrow[n \to \infty]{} \Phi(x)$$

Se n è "abbastanza grande",

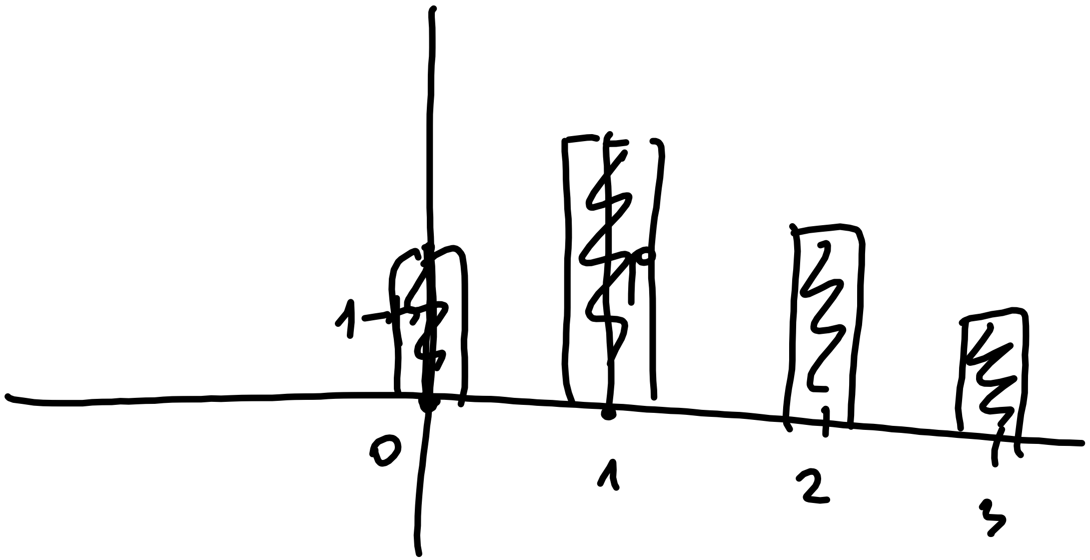
$$P(S_n^* \le x) \approx \Phi(x) = \Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right)$$

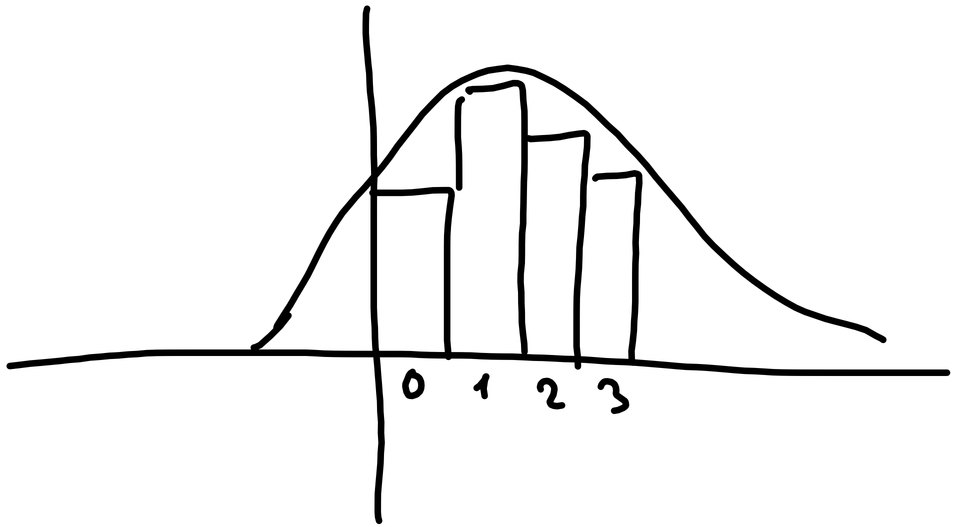
---

per n grande

$$P(S_n \le t) \approx \underbrace{\Phi\left(\frac{t - n\mu}{\sigma\sqrt{n}}\right)}$$

Approssimazione normale





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(Legge del χ^2 (chi-quadrato) a n gradi di libertà.

$\Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

Siano $X_1, X_2, \dots, X_n$ n v. a. indipendenti, tutte con legge $N(0,1)$

<u>Definizione</u>. Si chiama densità del χ^2 a n gradi di libertà ($\chi^2(n)$) la legge della v. a.

$$Y = X_1^2 + \dots + X_n^2$$

$$\chi^2(n) = \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$$

<u>Eserc.</u> Sia $X \sim N(0,1)$.

Calcolare la legge di $Y = X^2 = \varphi(X)$

$$P(Y \le t) = \begin{cases} 0 & \text{per } t \le 0 \\ & \text{per } t > 0 \end{cases}$$

Per $t > 0$ $\qquad x^2 \le 8$

$$P(Y \le t) = P(X^2 \le t) =$$

$$= P(-\sqrt{t} \le X \le \sqrt{t}) =$$

$$= \int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

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$$G(t) = P(Y \le t) = \begin{cases} 0 & t \le 0 \\ \displaystyle\int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx & t > 0 \end{cases}$$

$$G'(t) = \begin{cases} 0 & t \le 0 \\ \dfrac{1}{\sqrt{2\pi t}} e^{-t/2} & t > 0 \end{cases}$$

$$\frac{d}{dt}\left(\int_{-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \right) = \frac{d}{dt}\left(\int_0^{\sqrt{t}} \cdots + \int_{-\sqrt{t}}^{0} \cdots \right) =$$

$$= \frac{d}{dt}\left(\int_0^{\sqrt{t}} \cdots - \int_0^{-\sqrt{t}} \right) =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-t/2} \cdot \frac{1}{2\sqrt{t}} - \frac{1}{\sqrt{2\pi}} e^{-t/2} \left(-\frac{1}{2\sqrt{t}} \right)$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-t/2}$$

$$G'(t) = \begin{cases} 0 & t \le 0 \\ \frac{1}{\sqrt{2\pi t}} e^{-t/2} & t > 0 \end{cases} = \begin{cases} 0 & t \le 0 \\ \frac{1}{\sqrt{2\pi}} \underbrace{t^{-\frac{1}{2}} e^{-t/2}}_{\Gamma\left(\frac{1}{2}, \frac{1}{2}\right)} & t > 0 \end{cases}$$

$$= \begin{cases} 0 & t \le 0 \\ \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} \underbrace{t^{\frac{1}{2}-1} e^{-\frac{1}{2}t}} & t > 0 \end{cases}$$

$$\frac{\lambda^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t}$$

$$\frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2\pi}} \Bigg/ \frac{1}{\sqrt{2}\,\underbrace{\Gamma\left(\frac{1}{2}\right)}} = \frac{1}{\sqrt{2}\,\underbrace{\sqrt{\pi}}}$$

$$=$$

$Y = X_1^2 \qquad X_1 \sim N(0,1)$

$Y \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$

$Y = \underbrace{\underbrace{X_1^2}_{\Gamma\left(\frac{1}{2},\frac{1}{2}\right)} + \dots + \underbrace{X_n^2}_{\Gamma\left(\frac{1}{2},\frac{1}{2}\right)}}_{\sim \Gamma\left(\frac{n}{2},\frac{1}{2}\right)}$

somme di n v.a. indip. tutte con legge $\Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$

fie $Y \sim \chi^2(n) = \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

$\Gamma(\alpha, \lambda)$

$E[Y] = \frac{\alpha}{\lambda}$

$$E[Y] = \frac{\frac{n}{2}}{\frac{1}{2}} = n$$

$$E[Y] = E[X_1^2 + \dots + X_n^2] =$$

$$= \underbrace{E[X_1^2]}_{1} + \dots + \underbrace{E[X_n^2]}_{1} = n$$

$E[X_1]$

$X_1 \sim N(0,1)$

$$E[X_1^2] = E[X_1^2] - (E[X_1])^2 = \text{Var}\, X_1 = 1$$

Legge di Student a n gradi di libertà $t(n)$

Siano $X \sim N(0,1)$, $Y \sim \chi^2(n)$ } indipendenti

<u>Definizione</u>. Si chiama legge di Student a $\underline{\underline{n}}$ gr. di libertà la legge della v.a.

$$Z = \sqrt{n}\,\frac{X}{\sqrt{Y}}$$

Z e $-Z$ hanno la stessa legge

dunque Z è simmetrica

$$f(x) = f(-x)$$

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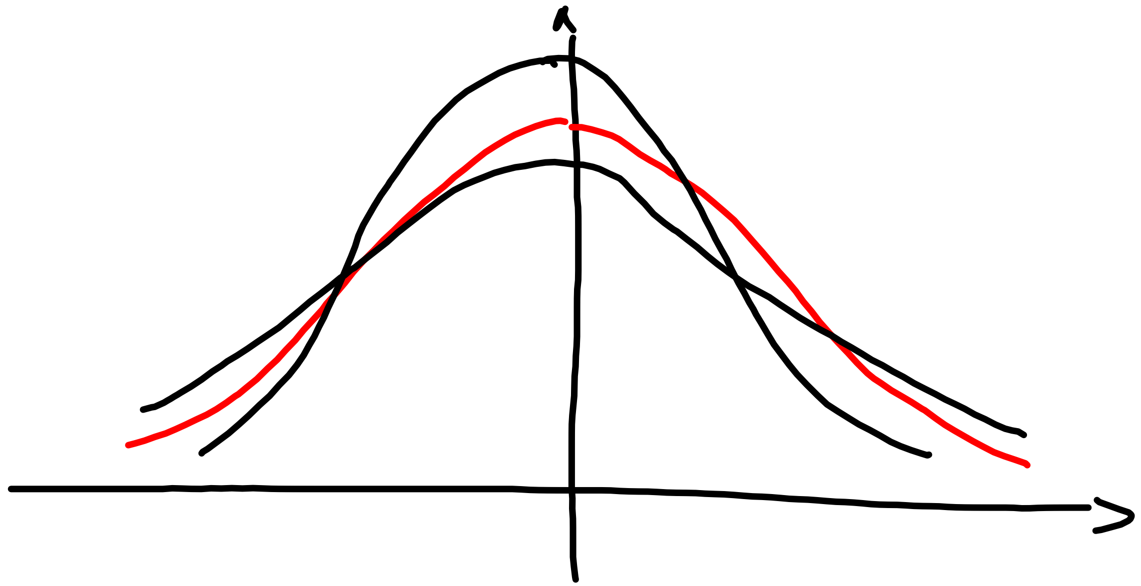

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Teorema di Cochran. Siano $X_1, X_2 \dots X_n$ n v.a. indipendenti, tutte con legge $N(\mu, \sigma^2)$.

Consideriamo le v.a.

$$\bar{X} = \frac{X_1 + \dots + X_n}{n} \quad ; \quad S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$$

✓ varianza camp

$$Z = \sqrt{n}\, \frac{\bar{X} - \mu}{\sigma} \quad ; \quad W = \frac{S^2 (n-1)}{\sigma^2} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sigma^2}$$

Allora:

1) $Z \sim N(0,1)$

2) $W \sim \chi^2(n-1)$

3) Z e W sono indipendenti.

$$W = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{\sigma^2}$$

$$\bar{X} = \frac{X_1 + \dots + X_n}{n} \approx \mu$$

$$\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 \sim \chi^2(n)$$

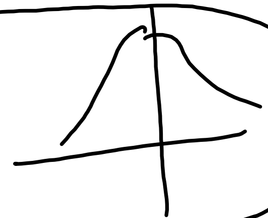
$$\frac{X_i - \mu}{\sigma} \sim N(0,1)$$

Corollario. La v. a.

$$T = \frac{\bar X - \mu}{S}\sqrt{n} \sim t(n-1)$$

$$\frac{\bar X - \mu}{\sigma}\sqrt{n} \sim N(0,1)$$

$$S^2 \approx \sigma^2$$



Dim.

$$T = \frac{\bar X - \mu}{S}\sqrt{n} = \frac{\bar X - \mu}{\sqrt{S^2}}\sqrt{n} =$$

$$= \frac{(\bar X - \mu)\sqrt{n}}{\sqrt{\frac{S^2(n-1)}{\sigma^2}\cdot\frac{\sigma^2}{n-1}}} = \frac{(\bar X - \mu)\sqrt{n}}{\sqrt{W}\,\frac{\sigma}{\sqrt{n-1}}} =$$

$$= \frac{\left(\frac{\bar X - \mu}{\sigma}\sqrt{n}\right)}{\sqrt{W}}\sqrt{n-1} = \frac{Z}{\sqrt{W}}\sqrt{n-1}$$

$Z \sim N(0,1)$

indépend.

$W \sim \chi^2(n-1)$

$$\sim t(n-1)$$

Fie $X \sim N(\underset{\mu}{1}, \underset{\sigma^2}{4})$

$P(X \le 1,2) =$ $\quad Y = \frac{X-1}{2} = \frac{x-\mu}{\sigma}$

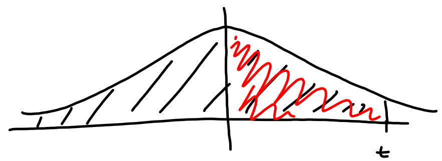
$= P\left(\frac{X-1}{2} \le \frac{1,2-1}{2}\right) = P(\underset{N(0,1)}{Y} \le 0,1)$

$= \phi(0,1) = 0,53983$

$\phi(0,17) = 0,56749$

$0,17 = \underline{\underline{0.1}} + \underline{\underline{0,007}}$

$$\phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$



$\phi(-0,17)$

$\phi(t) + \phi(-t) = 1$

$\phi(-t) = 1 - \phi(t)$

$\phi(-0.17) = 1 - \phi(0,17) =$

$= 1 - 0.56749 =$

$= 0.43251$

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Legge dei grandi numeri

Siano X una v.a. e

$(X_n)_n = X_1, X_2, X_3 \dots$ una successione di v.a.

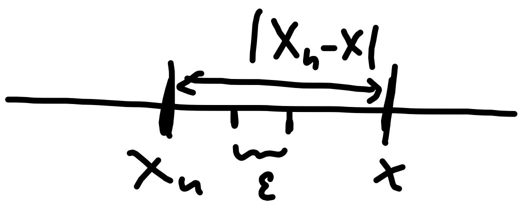
(a_n) $\{a_n\}$

<u>Definizione</u>. Si dice che (X_n) converge in probabilità verso X se

$\forall \varepsilon > 0$

$$\lim_{n \to \infty} P(|X_n - X| > \varepsilon) = 0$$

$$\boxed{X_n \xrightarrow{P} X}$$



Proposizione Sia (Z_n) una succ. di v. a. aventi tutte medie e varianze finite

Se accade che

(i) $\lim_n E[Z_n] = c \Leftarrow$

(ii) $\lim_n \mathrm{Var}\, Z_n = 0$

allora $Z_n \xrightarrow{P} c$

Dimostrazione.

Devo vedere che

$\forall\, \varepsilon > 0$

$$P(|Z_n - c| > \varepsilon) \longrightarrow 0$$

$$P(|Z_n - c| > \varepsilon) = P\left((Z_n - c)^2 > \varepsilon^2\right)$$

disug. di Markov

$$\leq \frac{E\left[(Z_n - c)^2\right]}{\varepsilon^2} \to 0$$

$$P(|Y| > x) \leq \frac{E[|Y|]}{x}$$

Markov

$$E\left[(Z_n - c)^2\right] \longrightarrow 0 \quad \text{Tesi.}$$

$$E\left[(Z_n - c)^2\right] =$$

$$= E\left[\left(\underbrace{\{Z_n - E[Z_n]\}}_{a} + \{\underbrace{E[Z_n]}_{b} - c\}\right)^2\right]$$

$$= E\left[\{Z_n - E[Z_n]\}^2 + \{\underline{E[Z_n] - c}\}^2 + \right.$$

$$\left. + 2\{Z_n - E[Z_n]\}\underline{\{E[Z_n] - c\}}\right]$$

$$= E\left[(Z_n - E[Z_n])^2\right] + (E[Z_n] - c)^2$$

$$+ 2(E[Z_n] - c)\underbrace{E[Z_n - E[Z_n]]}_{=0}$$

$$= \underbrace{\mathrm{Var}\, Z_n}_{\downarrow\, 0} + \underbrace{(E[Z_n] - c)^2}_{\downarrow\, 0}$$

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